Indian J. Phys. **47**, 390–395, (1973)

Radiation of a charged particle moving in a bounded cold plasma

Tushar Ray

Department of Physics, Panskura B. College, Panskura, Midanpore, W. Bengal

(*Received* 11 *July* 1972)

Fields generated by a helically moving charged particle in a cold magnetoplasma filling a cylindrical wave-guide have been calculated The radiation emitted by finite plasma modes is calculated

Introduction

Gyrating electron in free space has been treated by Montgomery & Tidman (1964) and in infinite anisotropic medium by McKenzie (1964). A similar problem concerned with radiation in a wave-guide filled with a uniform linear medium is discussed by Nakamura & Motz (1959). Recently Motz (1968) has applied finite transform method to the problem of helically moving electron in the cold magnetoplasma held in an infinitely long wave-guide of rectangular corress-section In the present paper the same problem is considered taking a circular cross-section of the wave-guide. Electric and magnetic fields of the oscillations produced are first calculated by travelling wave technique as applied by Parzen & Nomicos (1962). Of these oscillations there are certain modes which arise only on account of the finiteness of the plasma. The radiation emitted by these modes is then calculated.

Calculation

The wave-guide has circular cross-section of radius b and infinite length along z-axis of a cylindrical polar coordinate system. It is made of metal and filled with cold magnetoplasma which is characterized by di-electric tensor

$$\bar{\epsilon} \equiv \begin{pmatrix} \epsilon_1 & -i\epsilon_2 & 0 \\ i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}$$

where

$$\epsilon_1 = 1 - \frac{\omega_P^2}{\omega^2 - \omega_H^2}$$

$$\epsilon_2 = \frac{\omega}{\omega_H} \cdot \frac{\omega_P^2}{\omega^2 - \omega_H^2}$$

$$\epsilon_3 = 1 - \frac{\omega_P^2}{\omega^2}$$

ω_P and ω_H are the plasma and the cyclotron frequencies, respectively.

The current density J and the charge density ρ caused by a particle of charge e moving on a helix about the z-axis may be represented as

$$J_r = 0$$

$$J_\varphi = e\omega_0 \delta(r-a)\delta(z-ut)\delta(\phi-\omega_0 t)$$

$$J_z = eu\,\frac{\delta(r-a)}{a}\delta(z-ut)\delta(\phi-\omega_0 t)$$

$$\rho = e\frac{\delta(r-a)}{a} \cdot \delta(z-ut)\delta(\phi-\omega_0 t)$$

Here a is taken as the radius of the helix, and ω_0 and u are the angular and the axial velocities, respectively of the particle Using Maxwell's equations

$$\text{Curl}\,\boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{H}}{\partial t}; \quad \text{Curl}\,\boldsymbol{H} = \frac{1}{c} \cdot \frac{\partial}{\partial t}(\bar{\epsilon}.\boldsymbol{E}) + \frac{4\pi}{c}\boldsymbol{J}$$

$$\text{div}\,\boldsymbol{H} = 0, \qquad \text{div}\,(\bar{\epsilon}.\boldsymbol{E}) = 4\pi\rho.$$

we get for the z-components of $\boldsymbol{E}$ and $\boldsymbol{H}$

$$\mathrm{L}[E_z] + \frac{\epsilon_3}{\epsilon_1} \cdot \frac{\partial^2 E_z}{\partial z^2} + \frac{\omega^2}{c^2}\epsilon_3 E_z + \frac{\omega}{c} \cdot \frac{\epsilon_2}{\epsilon_1} \cdot \frac{\partial H_z}{\partial z} = \frac{4\pi}{\epsilon_1} \cdot \frac{\partial \rho}{\partial z} - \frac{4\pi i\omega}{c^2} \cdot J_z, \qquad \ldots \quad (1)$$

$$\mathrm{L}[H_z] + \frac{\partial^2 H_z}{\partial z^2} + \frac{\omega^2}{c^2\epsilon_1}(\epsilon_1^2 - \epsilon_2^2)H_z - \frac{\omega}{c} \cdot \frac{\epsilon_2\epsilon_3}{\epsilon_1} \cdot \frac{\partial E}{\partial z} = -\frac{4\pi}{c}[\boldsymbol{\nabla}\times\boldsymbol{J}]_z - \frac{4\pi\omega\epsilon_2\rho}{c\epsilon_1}. \quad \ldots \quad (2)$$

Here an $\exp(-i\omega t)$ time harmonic dependence is assumed and the operator L is defined as

$$\mathrm{L} \equiv \frac{\partial}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \phi^2}.$$

The current and charge densities may be written as

$$J_{\rho,\varphi,z} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}K_z J_{nr,\varphi,z} \cdot \exp i(K_z \cdot z + n\phi - \omega t)$$

$$\rho = \sum_{n=-\infty} \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}K_z\, \rho_n \cdot \exp i(K_z \cdot z + n\phi - \omega t)$$

where

$$J_{nr} = 0, \quad J_{n\varphi} = \frac{e\omega_0}{2\pi}\,\delta(r-a)\delta(\omega-n\omega_0-uK_z),$$

$$J_{nz} = \frac{eu}{2\pi a}\delta(r-a)\delta(\omega-n\omega_0-uK_z); \quad \rho_n = \frac{e}{2\pi a}\,\delta(r-a)\delta(\omega-n\omega_0-uK_z).$$

These forms of the current and charge densities suggest that $\boldsymbol{E}$ and $\boldsymbol{H}$ may be written as

$$\left.\begin{aligned} E_{r,\varphi,z} &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dK_z E_{n\rho,\varphi,z} \exp i(K_z.\,\mathbf{z}+n\phi-\omega t), \\ H_{\rho,\varphi,z} &= \sum_{n=\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dK_z H_{n\rho,\varphi,z}\,.\, \exp i(K_z.\,\mathbf{z}+n\phi-\omega t). \end{aligned}\right\} \quad \ldots (3)$$

Inspection of equations (1) and (2) indicates that the solutions for E_n and H_n should have the forms

$$\left.\begin{aligned} E_{nz} &= A\; J_n(\lambda r) && \text{for } r < a \\ &= B.J_n(\lambda r)+C.H_n^{(2)}(\lambda r) && \text{for } b > r > a \\ H_{nz} &= D.J_n(\lambda r) && \text{for } r < a \\ &= E.J_n(\lambda r)+F\; H_n^{(2)}(\lambda r) && \text{for } b > r > a \end{aligned}\right\} \quad \ldots (4)$$

where J_n and $H_n^{(2)}$ are the Bessel and Hankel functions, respectively. λ is given by the equation :

$$\begin{vmatrix} \lambda^2+\dfrac{K_z^2\epsilon_3}{\epsilon_1}-\dfrac{\omega^2}{c^2}\epsilon_3 & -\dfrac{iK_z\omega}{c}\;\dfrac{\epsilon_2}{\epsilon_1} \\ \dfrac{iK_z\omega}{c}\,.\,\dfrac{\epsilon_2\epsilon_3;}{\epsilon_1} & \lambda^2+K_z^2-\dfrac{\omega^2}{c^2\epsilon_1}(\epsilon_1^2-\epsilon_2^2) \end{vmatrix} = 0 \quad \ldots (5)$$

From Maxwell's equations the other components of $\boldsymbol{E}$ and $\boldsymbol{H}$ may be expressed in terms of E_z and H_z and we get

$$E_{nr} = \frac{\omega/c}{\Delta}\left[-i\epsilon_2\frac{ncK_z}{\omega r}E_{nz}-\epsilon_1'.\frac{n}{r}H_{nz}+i\epsilon_1'\frac{cK_z}{\omega}\frac{\partial E_{nz}}{\partial r}+\epsilon_2\frac{\partial H_{nz}}{\partial r}\right]$$

$$E_{n\varphi} = \frac{\omega/c}{\Delta}\left[-\epsilon_1'\frac{ncK_z}{\omega r}E_{nz}+i\epsilon_2\frac{n}{r}H_{nz}+\frac{\epsilon_2 cK_z}{\omega}\frac{\partial E_{nz}}{\partial r}-i\epsilon_1'\frac{\partial H_{nz}}{\partial r}\right]$$

$$H_{nr} = \frac{nc}{r\omega}E_{nz}-\frac{K_z}{\Delta}\left[-\epsilon_1'\frac{ncK_z}{\omega r}E_{nr}+i\epsilon_2\frac{n}{r}H_{nz}+\epsilon_2\frac{cK_z}{\omega}\frac{\partial E_{nz}}{\partial r}-i\epsilon_1'\frac{\partial H_{nz}}{\partial r}\right]$$

$$H_{n\varphi} = \frac{ic}{\omega}.\frac{\partial E_{nz}}{\partial r}+\frac{K_z}{\Delta}\left[-i\epsilon_2\frac{ncK_z}{\omega r}E_{nz}-\epsilon_1'\frac{n}{r}H_{nz}+i\epsilon_1'\frac{cK_z}{\omega}\frac{\partial E_{nz}}{\partial r}+\epsilon_2\frac{\partial H_{nz}}{\partial r}\right]$$

where

$$\Delta = \frac{\omega^2}{c^2}(\epsilon_1'^2 - \epsilon_2^2); \qquad \epsilon_1' = \epsilon_1 - \left(\frac{cK_z}{\omega}\right)^2.$$

The constants A, B, C, D, E and F in equation (4) are calculated from the matching of the tangential fields at $r = a$ with the proper jump conditions and from the boundary conditions at $r = b$

These are

$$A = \frac{\pi e}{c\epsilon_1}\left[\frac{\epsilon_1' u\omega}{c} + \frac{ncK_z\omega_0}{\omega}\right]. \frac{H_n^{(2)}(\lambda b)J_n(\lambda a) - J_n(\lambda b)H_n^{(2)}(\lambda a)}{J_n(\lambda b)}.$$
$$\times\, \delta(\omega - n\omega_0 - uK_z)$$

$$B = A. \frac{J_n(\lambda a).H_n^{(2)}(\lambda b)}{H_n^{(2)}(\lambda b)J_n(\lambda a) - J_n(\lambda b)H_n^{(2)}(\lambda a)}.$$

$$C = -B.\frac{J_n(\lambda b)}{H_n^{(2)}(\lambda b)}$$

$$D = \frac{\pi i e\epsilon_2}{c\epsilon_1}(uK_z - n\omega_0). \frac{H_n^{(2)}(\lambda a)J_n'(\lambda b) - H_n^{(2)}(\lambda b)J_n(\lambda a)}{J_n'(\lambda b)}.$$
$$\times\, \delta(\omega - n\omega_0 - uK_z) \tag{6}$$
$$+ \frac{\pi i e\omega_0 a\lambda}{c}. \frac{J_n'(\lambda b)H_n'^{(2)}(\lambda a) - H_n'^{(2)}(\lambda b)J_n'(\lambda a)}{J_n'(\lambda b)}. \delta(\omega - n\omega_0 - uK_z)$$

$$E = \frac{\pi i e\epsilon_2}{c\epsilon_1}(n\omega_0 - uK_z). \frac{J_n(\lambda a)H_n'^{(2)}(\lambda b)}{J_n'(\lambda b)}\ \delta(\omega - n\omega_0 - uK_z)$$
$$- \frac{\pi i e\omega_0 a\lambda}{c}. \frac{J_n'(\lambda a)H_n'^{(2)}(\lambda b)}{J_n'(\lambda b)}. \delta(\omega - n\omega_0 - uK_z)$$

$$F = \frac{\pi i e}{c}\left[\frac{\epsilon_2}{\epsilon_1}(uK_z - n\omega_0)J_n(\lambda a) + a\omega_0\lambda J_n'(\lambda a)\right]. \delta(\omega - n\omega_0 - uK_z)$$

where

$$J_n'(\lambda r) = \frac{\partial J_n(\lambda r)}{\partial(\lambda r)} \qquad \text{and} \qquad H_n'^{(2)}(\lambda r) = \frac{\partial H_n^{(2)}(\lambda r)}{\partial(\lambda r)}.$$

In the expressions (3) for the fields, the integrations with respect to ω is very simple because of the delta function in (5). For the integration with respect to K_z we see that the contributions to the integral come from the poles (a) $\Delta = 0$, (b) $J_n(\lambda b) = 0$ and (c) $J_n'(\lambda b) = 0$. The poles in (a) give rise to infinite plasma modes and those in (b) and (c) to modes which depend on the radius of the plasma cylinder. For the poles in (b)

$$J_n(\lambda b) = 0$$

or let $\lambda b = \lambda_{nm}.b, \quad m = 1, 2, 3, \ldots$

With λ replaced by λ_{nm}, the equation (5) yields several values of K_z. Of these only the real ones are to be considered for the oscillations to be sustained. Let these real solutions be denoted by K_{znm}. There may be more than one such solutions for each value of λ_{nm}. Similarly, for the poles in (c), let the real values of K_z be denoted by K_{znm} and the corresponding value of λ be λ_{nm}'. For $(Z-ut) > 0$, we choose the integration path along the upper semi-circle in the K_z-plane and the lower semi-circle for $(Z-ut) < 0$. For outgoing waves, either the positive or the negative values of K_{znm} or K_{znm}' need be included in the contour

The radiation per unit time through the cross-section of the wave-guide is given by

$$S = \frac{c}{4\pi} \int_{r=0}^{b} \int_{\varphi=0}^{2\pi} (E_r H_\varphi - H_r E_\varphi) r dr d\phi$$

Making use of the relation

$$2 \int_0^b r dr \ J_n'^2(\lambda r) + \frac{n^2}{r^2\lambda^2} \ J_n^2(\lambda r) \ = b^2 J_n'^2(\lambda b) \qquad \text{when } J_n(\lambda b) = 0$$

$$= \left(b^2 - \frac{n^2}{\lambda^2} \right) J_n^2(\lambda b) \ \text{ when } J_n'(\lambda b) = 0,$$

we get after some simplifications

$$S = \sum_{n=1}^{\infty} S_n$$

with

$$S_n = \sum_m \frac{\pi^3 e^2}{\epsilon_1^2} \left(\frac{\epsilon'_{1nm} u}{c} + \frac{nc K_{znm} \omega_0}{\Omega^2} \right) . \frac{K_{znm} \lambda^2_{nm}}{\Delta_n} .$$

$$\left\{ \frac{\epsilon_2^2 K^2_{znm}}{\Delta_{nm}} + \left(1 + \frac{\epsilon'_{1nm} K^2_{znm}}{\Delta_n} \right) \epsilon'_{1nm} \right\} \times \left[\frac{H_n^{(2)}(\lambda_{nm} b) J_n(\lambda_{nm} a)}{\left| \dfrac{d\lambda}{dK_z} \right|_{nm}} \right]^2 .$$

$$+ \sum_{m'} \frac{\pi^3 e^2}{c^2} \cdot \frac{\Omega' K_{znm}'}{\Delta_n} . (\epsilon'_{1nm}' + \epsilon_2^2) . \left\{ \frac{\epsilon_2}{\epsilon_1} (u K_{znm}' - n\omega_0) J_n(\lambda_{nm}' a) + \right.$$

$$\left. + a\omega_0 \lambda_{nm}' J_n'(\lambda_{nm}' a) \right\}^2 \left[\frac{H_n^{(2)\prime}(\lambda_{nm}' b) J_n(\lambda_{nm}' b)}{\left| \dfrac{d\lambda}{dK_z} \right|_{nm'} . J_n''(\lambda_{nm}' b)} \right]^2 . \ \left(\lambda^2_{nm}' - \frac{n^2}{b^2} \right).$$

where $\Omega = n\omega_0+uK_{znm}$ and $\Omega' = n\omega_0+uK_{znm}'$. The subscripts nm and nm' denote the quantities in which K_z is replaced by K_{znm} and K_{znm}', respectively and λ by λ_{nm} and λ_{nm}'. The values of ϵ_1, ϵ_2 and ϵ_3 are used with ω replaced by Ω in the first term and by Ω' in the second. And from the equation (5) we get

$$\frac{d\lambda}{K_z} = \frac{\dfrac{K_z}{\lambda}\cdot 2\dfrac{\Omega^2}{c^2}\cdot \dfrac{\epsilon_3\epsilon_1'}{\epsilon_1} - \lambda^2\left(1+\dfrac{\epsilon_3}{\epsilon_1}\right)}{2\lambda^2-\dfrac{\omega^2}{c^2}\left\{\epsilon_1'\left(1+\dfrac{\epsilon_3}{\epsilon_1}\right)-\dfrac{\epsilon_2^2}{\epsilon_1}\right\}}$$

The summations over m and m' extend over all the zeroes of $J_n(\lambda b)$ and $J_n'(\lambda b)$ respectively which are included in the contour of integration

Acknowledgement

The author expresses his deep gratitude to Dr. M R Gupta for suggesting the problem and for his valuable help in its solution.

References

Mckenzie J. F. 1964 *Proc. Phys. Soc. London* **84**, 269.
Montgomery D. C. & Tidman D. A. 1964 *Plasma Kinetic Theory* McGraw-Hill Book Co. p. 212.
Motz H. 1968 *Acta Phys. Aust.* **28**, 139.
Nakamura M. & Motz H. 1959 *Ann. of Phys.* **7**, 84.
Parzen P. & Nomicos G. 1962 *J. Math. Phys.* **3**,